

# HEAT TRACES AND SPECTRAL ZETA FUNCTIONS FOR $p$ -ADIC LAPLACIANS

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**ABSTRACT.** In this article we initiate the study of the heat traces and spectral zeta functions for certain  $p$ -adic Laplacians. We show that the heat traces are given by  $p$ -adic integrals of Laplace type, and that the spectral zeta functions are  $p$ -adic integrals of Igusa-type. We find good estimates for the behaviour of the heat traces when the time tends to infinity, and for the asymptotics of the function counting the eigenvalues less than or equal to a given quantity.

## 1. INTRODUCTION

The  $p$ -adic heat equation is defined as

$$(1.1) \quad \frac{\partial u(x, t)}{\partial t} + D^\beta u(x, t) = 0, \quad x \in \mathbb{Q}_p, \quad t \geq 0$$

where

$$(D^\beta \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left( |\xi|_p^\beta \mathcal{F}_{x \rightarrow \xi} \varphi \right), \quad \beta > 0,$$

is the Vladimirov operator (a  $p$ -adic Laplacian), and  $\mathcal{F}$  denotes the  $p$ -adic Fourier transform. This equation is the  $p$ -adic counterpart of the classical fractional heat equation, which describes particle performing a random motion (the fractional Brownian motion), a ‘similar’ statement is valid for the  $p$ -adic heat equation. More precisely, the fundamental solution of (1.1) is the transition density of a bounded right-continuous Markov process without second kind discontinuities. The family of non-Archimedean heat-type equations is very large and it has deep connections with mathematical physics. For instance, in [4]-[5], Avetisov et al. introduced a new class of models for complex systems based on  $p$ -adic analysis. From a mathematical point view, in these models the time-evolution of a complex system is described by a  $p$ -adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a Markov process on an ultrametric space. The simplest type of master equation is the one-dimensional  $p$ -adic heat equation. This equation was introduced in the book of Vladimirov, Volovich and Zelenov [32]. It is worth to mention here, that the  $p$ -adic heat equation also appeared in certain works connected with the Riemann hypothesis [20]. In recent years the non-Archimedean heat-type equations and their associated Markov processes have been studied intensively, see e.g. [18], [32], [7], [9], [10], [15], [27], [31], [35], [36] and the references therein.

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The connections between the Archimedean heat equations with number theory and geometry are well-known and deep. Let us mention here, the connection with the Riemann zeta function which drives naturally to trace-type formulas, see e.g. [3, and the references therein], and the connection with the Atiyah-Singer index Theorem, see e.g. [16, and the references therein]. The study of non-Archimedean counterparts of the above mentioned matters is quite relevant, specially taking into account that the Connes and Deninger programs to attack the Riemann hypothesis drive naturally to these matters, see e.g. [12], [13], [21] and the references therein. For instance several types of  $p$ -adic trace formulas have been studied, see e.g. [1], [6], [34] and the references therein.

Nowadays there is no a theory of pseudodifferential operators over  $p$ -adic manifolds comparable to the classical theory, see e.g. [28] and the references therein. The  $n$ -dimensional unit ball is the simplest  $p$ -adic compact manifold possible. From a topological point of view this ball is a fractal, more precisely, it is topologically equivalent to a Cantor-like subset of the real line, see e.g. [2], [32]. Currently, there is a lot interest on spectral zeta functions attached to fractals see e.g. [19], [29].

In this article we initiate the study of heat traces and spectral zeta functions attached to certain  $p$ -adic Laplacians, denoted as  $\mathbf{A}_\beta$ , which are generalizations of the  $p$ -adic Laplacians introduced by the authors in [9], see also [10]. By using an approach inspired on the work of Minakshisundaram and Pleijel, see [22]-[24], we find a formula for the trace of the semigroup  $e^{-t\mathbf{A}_\beta}$  acting on the space of square integrable functions supported on the unit ball with average zero, see Theorem 6.7. The trace of  $e^{-t\mathbf{A}_\beta}$  is a  $p$ -adic oscillatory integral of Laplace-type, we do not know the exact asymptotics of this integral as  $t$  tends to infinity, however, we obtain a good estimation for its behavior at infinity, see Theorem 6.7 (ii). Several unexpected mathematical situations occur in the  $p$ -adic setting. For instance, the spectral zeta functions are  $p$ -adic Igusa-type integrals, see Theorem 7.5. The  $p$ -adic spectral zeta functions studied here may have infinitely many poles on the boundary of its domain of holomorphy, then, to the best of our knowledge, the standard Ikehara Tauberian Theorems cannot be applied to obtain the asymptotic behavior for the function counting the eigenvalues of  $\mathbf{A}_\beta$  less than or equal to  $T \geq 0$ , however, we are still able to find good estimates for this function, see Theorem 7.5 and Remark 7.6, and Conjecture 7.7. The proofs require several results on certain ‘boundary value problems’ attached to  $p$ -adic heat equations associated with operators  $\mathbf{A}_\beta$ , see Proposition 5.3, Theorem 6.5 and Proposition 6.6. Finally, Let us mention that our results and techniques are completely different from those presented in [1], [6], [34].

## 2. PRELIMINARIES

In this section we fix the notation and collect some basic results on  $p$ -adic analysis that we will use through the article. For a detailed exposition on  $p$ -adic analysis the reader may consult [2], [30], [32].

**2.1. The field of  $p$ -adic numbers.** Along this article  $p$  will denote a prime number. The field of  $p$ -adic numbers  $\mathbb{Q}_p$  is defined as the completion of the field of rational numbers  $\mathbb{Q}$  with respect to the  $p$ -adic norm  $|\cdot|_p$ , which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b}, \end{cases}$$

where  $a$  and  $b$  are integers coprime with  $p$ . The integer  $\gamma = \text{ord}_p(x) := \text{ord}(x)$ , with  $\text{ord}(0) := +\infty$ , is called the  $p$ -adic order of  $x$ . We extend the  $p$ -adic norm to  $\mathbb{Q}_p^n$  by taking

$$\|x\|_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

We define  $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$ , then  $\|x\|_p = p^{-\text{ord}(x)}$ . The metric space  $(\mathbb{Q}_p^n, \|\cdot\|_p)$  is a complete ultrametric space. As a topological space  $\mathbb{Q}_p$  is homeomorphic to a Cantor-like subset of the real line, see e.g. [2], [32].

Any  $p$ -adic number  $x \neq 0$  has a unique expansion of the form

$$x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j,$$

where  $x_j \in \{0, 1, 2, \dots, p-1\}$  and  $x_0 \neq 0$ . By using this expansion, we define the fractional part  $\{x\}_p$  of  $x \in \mathbb{Q}_p$  as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\ p^{\text{ord}(x)} \sum_{j=0}^{-\text{ord}(x)-1} x_j p^j & \text{if } \text{ord}(x) < 0. \end{cases}$$

In addition, any  $x \in \mathbb{Q}_p^n \setminus \{0\}$  can be represented uniquely as  $x = p^{\text{ord}(x)} v(x)$  where  $\|v(x)\|_p = 1$ .

**2.2. Additive characters.** Set  $\chi_p(y) = \exp(2\pi i \{y\}_p)$  for  $y \in \mathbb{Q}_p$ . The map  $\chi_p(\cdot)$  is an additive character on  $\mathbb{Q}_p$ , i.e. a continuous map from  $(\mathbb{Q}_p, +)$  into  $S$  (the unit circle considered as multiplicative group) satisfying  $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$ ,  $x_0, x_1 \in \mathbb{Q}_p$ . The additive characters of  $\mathbb{Q}_p$  form an Abelian group which is isomorphic to  $(\mathbb{Q}_p, +)$ , the isomorphism is given by  $\xi \rightarrow \chi_p(\xi x)$ , see e.g. [2, Section 2.3].

**2.3. Topology of  $\mathbb{Q}_p^n$ .** For  $r \in \mathbb{Z}$ , denote by  $B_r^n(a) = \{x \in \mathbb{Q}_p^n; \|x - a\|_p \leq p^r\}$  the ball of radius  $p^r$  with center at  $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$ , and take  $B_r^n(0) := B_r^n$ . Note that  $B_r^n(a) = B_r(a_1) \times \dots \times B_r(a_n)$ , where  $B_r(a_i) := \{x \in \mathbb{Q}_p; |x - a_i|_p \leq p^r\}$  is the one-dimensional ball of radius  $p^r$  with center at  $a_i \in \mathbb{Q}_p$ . The ball  $B_0^n$  equals the product of  $n$  copies of  $B_0 = \mathbb{Z}_p$ , the ring of  $p$ -adic integers. We also denote by  $S_r^n(a) = \{x \in \mathbb{Q}_p^n; \|x - a\|_p = p^r\}$  the sphere of radius  $p^r$  with center at  $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$ , and take  $S_r^n(0) := S_r^n$ . We notice that  $S_0^1 = \mathbb{Z}_p^\times$  (the group of units of  $\mathbb{Z}_p$ ), but  $(\mathbb{Z}_p^\times)^n \subsetneq S_0^n$ . The balls and spheres are both open and closed subsets in  $\mathbb{Q}_p^n$ . In addition, two balls in  $\mathbb{Q}_p^n$  are either disjoint or one is contained in the other.

As a topological space  $(\mathbb{Q}_p^n, \|\cdot\|_p)$  is totally disconnected, i.e. the only connected subsets of  $\mathbb{Q}_p^n$  are the empty set and the points. A subset of  $\mathbb{Q}_p^n$  is compact if and only if it is closed and bounded in  $\mathbb{Q}_p^n$ , see e.g. [32, Section 1.3], or [2, Section 1.8]. The balls and spheres are compact subsets. Thus  $(\mathbb{Q}_p^n, \|\cdot\|_p)$  is a locally compact topological space.

We will use  $\Omega(p^{-r}\|x - a\|_p)$  to denote the characteristic function of the ball  $B_r^n(a)$ . For more general sets, we will use the notation  $1_A$  for the characteristic function of a set  $A$ .

### 3. THE BRUHAT-SCHWARTZ SPACE AND THE FOURIER TRANSFORM

A complex-valued function  $\varphi$  defined on  $\mathbb{Q}_p^n$  is called *locally constant* if for any  $x \in \mathbb{Q}_p^n$  there exist an integer  $l(x) \in \mathbb{Z}$  such that

$$(3.1) \quad \varphi(x + x') = \varphi(x) \text{ for } x' \in B_{l(x)}^n.$$

A function  $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$  is called a *Bruhat-Schwartz function* (or a *test function*) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The  $\mathbb{C}$ -vector space of Bruhat-Schwartz functions is denoted by  $\mathcal{D}(\mathbb{Q}_p^n)$ . For  $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ , the largest number  $l = l(\varphi)$  satisfying (3.1) is called *the exponent of local constancy* (or *the parameter of constancy*) of  $\varphi$ .

If  $U$  is an open subset of  $\mathbb{Q}_p^n$ ,  $\mathcal{D}(U)$  denotes the space of test functions with supports contained in  $U$ , then  $\mathcal{D}(U)$  is dense in

$$L^\rho(U) = \left\{ \varphi : U \rightarrow \mathbb{C}; \|\varphi\|_\rho = \left\{ \int_U |\varphi(x)|^\rho d^n x \right\}^{\frac{1}{\rho}} < \infty \right\},$$

where  $d^n x$  is the Haar measure on  $\mathbb{Q}_p^n$  normalized by the condition  $\text{vol}(B_0^n) = 1$ , for  $1 \leq \rho < \infty$ , see e.g. [2, Section 4.3].

**3.1. The Fourier transform of test functions.** Given  $\xi = (\xi_1, \dots, \xi_n)$  and  $y = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$ , we set  $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$ . The Fourier transform of  $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$  is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x) \varphi(x) d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where  $d^n x$  is the normalized Haar measure on  $\mathbb{Q}_p^n$ . The Fourier transform is a linear isomorphism from  $\mathcal{D}(\mathbb{Q}_p^n)$  onto itself satisfying  $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$ , see e.g. [2, Section 4.8]. We will also use the notation  $\mathcal{F}_{x \rightarrow \xi} \varphi$  and  $\widehat{\varphi}$  for the Fourier transform of  $\varphi$ .

### 4. A CLASS OF $p$ -ADIC LAPLACIANS

Take  $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$ , and fix a function

$$A : \mathbb{Q}_p^n \rightarrow \mathbb{R}_+$$

satisfying the following properties:

- (i)  $A(\xi)$  is a radial function, i.e.  $A(\xi) = g(\|\xi\|_p)$  for some  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , by simplicity we use the notation  $A(\xi) = A(\|\xi\|_p)$ ;
- (ii) there exist constants  $C_0, C_1 > 0$  and  $\beta > 0$  such that

$$(4.1) \quad C_0 \|\xi\|_p^\beta \leq A(\xi) \leq C_1 \|\xi\|_p^\beta, \text{ for } \xi \in \mathbb{Q}_p^n.$$

Taking into account that  $\beta$  in (4.1) is unique, we use the notation  $A_\beta(\|\xi\|_p) = A(\|\xi\|_p)$ .

We define the pseudodifferential operator  $\mathbf{A}_\beta$  by

$$(4.2) \quad (\mathbf{A}_\beta \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} [A_\beta(\xi) \mathcal{F}_{x \rightarrow \xi} \varphi], \text{ for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

We will call  $A_\beta(\xi)$  the symbol of  $\mathbf{A}_\beta$ . The operator  $\mathbf{A}_\beta$  extends to an unbounded and densely defined operator in  $L^2(\mathbb{Q}_p^n)$  with domain

$$(4.3) \quad \text{Dom}(\mathbf{A}_\beta) = \{\varphi \in L^2; A_\beta(\xi)\mathcal{F}\varphi \in L^2\}.$$

In addition:

- (i)  $(\mathbf{A}_\beta, \text{Dom}(\mathbf{A}_\beta))$  is self-adjoint and positive operator;
- (ii)  $-\mathbf{A}_\beta$  is the infinitesimal generator of a contraction  $C_0$ -semigroup, cf. [9, Proposition 3.3].

We attach to operator  $\mathbf{A}_\beta$  the following ‘heat equation’:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \mathbf{A}_\beta u(x,t) = 0, & x \in \mathbb{Q}_p^n, \quad t \in [0, \infty) \\ u(x,0) = u_0(x), & u_0(x) \in \text{Dom}(\mathbf{A}_\beta). \end{cases}$$

This initial value problem has a unique solutions given by

$$u(x,t) = \int_{\mathbb{Q}_p^n} Z(x-y,t) u_0(y) d^n y,$$

where

$$Z(x,t; \mathbf{A}_\beta) := Z(x,t) = \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) e^{-tA_\beta(\xi)} d^n \xi, \quad \text{for } t > 0, \quad x \in \mathbb{Q}_p^n,$$

cf. [9, Theorem 6.5]. The function  $Z(x,t)$  is called the heat kernel associated with operator  $\mathbf{A}_\beta$ .

**4.1. Operators  $\mathbf{W}_\alpha$ .** The class of operators  $\mathbf{A}_\beta$  includes the class of operators  $\mathbf{W}_\alpha$  studied by the authors in [9], see also [10]. In addition, most of the results on  $\mathbf{W}_\alpha$  operators are valid for  $\mathbf{A}_\beta$  operators. We review briefly the definition of these operators. Fix a function

$$w_\alpha : \mathbb{Q}_p^n \rightarrow \mathbb{R}_+$$

satisfying the following properties:

- (i)  $w_\alpha(y)$  is a radial i.e.  $w_\alpha(y) = w_\alpha(\|y\|_p)$ ;
- (ii)  $w_\alpha(\|y\|_p)$  is continuous and increasing function of  $\|y\|_p$ ;
- (iii)  $w_\alpha(y) = 0$  if and only if  $y = 0$ ;
- (iv) there exist constants  $C_0, C_1 > 0$  and  $\alpha > n$  such that

$$C_0 \|y\|_p^\alpha \leq w_\alpha(\|y\|_p) \leq C_1 \|y\|_p^\alpha, \quad \text{for } y \in \mathbb{Q}_p^n.$$

We now define the operator

$$(\mathbf{W}_\alpha \varphi)(x) = \kappa \int_{\mathbb{Q}_p^n} \frac{\varphi(x-y) - \varphi(x)}{w_\alpha(\|y\|_p)} d^n y, \quad \text{for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n),$$

where  $\kappa$  is a positive constant. The operator  $\mathbf{W}_\alpha$  is pseudodifferential, more precisely, if

$$A_{w_\alpha}(\xi) := \int_{\mathbb{Q}_p^n} \frac{1 - \chi_p(y \cdot \xi)}{w_\alpha(\|y\|_p)} d^n y,$$

then

$$(\mathbf{W}_\alpha \varphi)(x) = -\kappa \mathcal{F}_{\xi \rightarrow x}^{-1} [A_{w_\alpha}(\xi) \mathcal{F}_{x \rightarrow \xi} \varphi], \quad \text{for } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).$$

The function  $A_{w_\alpha}(\xi)$  is radial (so we use the notations  $A_{w_\alpha}(\xi) = A_{w_\alpha}(\|\xi\|_p)$ ), continuous, non-negative,  $A_{w_\alpha}(0) = 0$ , and it satisfies

$$C'_0 \|\xi\|_p^{\alpha-n} \leq A_{w_\alpha}(\|\xi\|_p) \leq C'_1 \|\xi\|_p^{\alpha-n}, \text{ for } x \in \mathbb{Q}_p^n,$$

cf. [9, Lemmas 3.1, 3.2, 3.3]. The operator  $\mathbf{W}_\alpha$  extends to an unbounded and densely defined operator in  $L^2(\mathbb{Q}_p^n)$ .

#### 4.2. Examples.

**Example 4.1.** *The Taibleson operator is defined as*

$$\left(D_T^\beta \phi\right)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left( \|\xi\|_p^\beta \mathcal{F}_{x \rightarrow \xi} \phi \right), \text{ with } \beta > 0 \text{ and } \phi \in \mathcal{D}(\mathbb{Q}_p^n).$$

cf. [27], [2, Section 9.2.2].

**Example 4.2.** *Take  $A_\beta(\xi) = \|\xi\|_p^\beta \left\{ B - A e^{-\|\xi\|_p} \right\}$  with  $B > A > 0$ . Then  $A_\beta(\xi)$  satisfies all the requirements announced at the beginning of this section. In general, if  $f : \mathbb{Q}_p^n \rightarrow \mathbb{R}_+$  is a radial function satisfying*

$$0 < \inf_{\xi \in \mathbb{Q}_p^n} f(\|\xi\|_p) < \sup_{\xi \in \mathbb{Q}_p^n} f(\|\xi\|_p) < \infty,$$

*then  $A_\beta(\|\xi\|_p) f(\|\xi\|_p)$  satisfies all the requirements announced at the beginning of this section.*

#### 5. LIZORKIN SPACES, EIGENVALUES AND EIGENFUNCTIONS FOR $\mathbf{A}_\beta$ OPERATORS

We set  $\mathcal{L}_0(\mathbb{Q}_p^n) := \{\varphi \in \mathcal{D}(\mathbb{Q}_p^n); \widehat{\varphi}(0) = 0\}$ . The  $\mathbb{C}$ -vector space  $\mathcal{L}_0$  is called the  $p$ -adic Lizorkin space of second class. We recall that  $\mathcal{L}_0$  is dense in  $L^2$ , cf. [2, Theorem 7.4.3], and that  $\varphi \in \mathcal{L}_0(\mathbb{Q}_p^n)$  if and only if

$$(5.1) \quad \int_{\mathbb{Q}_p^n} \varphi(x) d^n x = 0.$$

Consider the operator  $(\mathbf{A}_\beta \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} [A_\beta(\xi) \mathcal{F}_{x \rightarrow \xi} \varphi]$  on  $\mathcal{L}_0(\mathbb{Q}_p^n)$ , then  $\mathbf{A}_\beta$  is densely defined on  $L^2$ , and  $\mathbf{A}_\beta : \mathcal{L}_0(\mathbb{Q}_p^n) \rightarrow \mathcal{L}_0(\mathbb{Q}_p^n)$  is a well-defined linear operator.

We set  $\mathcal{L}_0(\mathbb{Z}_p^n) := \{\varphi \in \mathcal{L}_0(\mathbb{Q}_p^n); \text{supp } \varphi \subseteq \mathbb{Z}_p^n\}$ , and define

$$L_0^2(\mathbb{Z}_p^n, d^n x) := L_0^2(\mathbb{Z}_p^n) = \left\{ f \in L^2(\mathbb{Z}_p^n, d^n x); \int_{\mathbb{Z}_p^n} f(x) d^n x = 0 \right\}.$$

Notice that, since  $L_0^2(\mathbb{Z}_p^n)$  is the orthogonal complement in  $L^2(\mathbb{Z}_p^n)$  of the space generated by the characteristic function of  $\mathbb{Z}_p^n$ , then  $L_0^2(\mathbb{Z}_p^n)$  is a Hilbert space.

Then  $\mathcal{L}_0(\mathbb{Z}_p^n)$  is dense in  $L_0^2(\mathbb{Z}_p^n)$ . Indeed, set

$$\delta_k(x) := p^{nk} \Omega(p^k \|x\|_p), \text{ for } k \in \mathbb{N}.$$

Then  $\int_{\mathbb{Q}_p^n} \delta_k(x) d^n x = 1$  for any  $k$ , and take  $f \in L_0^2(\mathbb{Z}_p^n)$ , then  $f_k = f * \delta_k \in \mathcal{L}_0(\mathbb{Z}_p^n)$ , and  $f_k \xrightarrow{\|\cdot\|_{L^2}} f$ .

Set

$$\omega_{\gamma bk}(x) := p^{-\frac{n\gamma}{2}} \chi_p(p^{-1}k \cdot (p^\gamma x - b)) \Omega(\|p^\gamma x - b\|_p),$$

where  $\gamma \in \mathbb{Z}$ ,  $b \in (\mathbb{Q}_p/\mathbb{Z}_p)^n$ ,  $k = (k_1, \dots, k_n)$  with  $k_i \in \{0, \dots, p-1\}$  for  $i = 1, \dots, n$ , and  $k \neq (0, \dots, 0)$ .

**Lemma 5.1.** *With the above notation,*

$$(\mathbf{A}_\beta \omega_{\gamma bk})(x) = \lambda_{\gamma bk} \omega_{\gamma bk}(x)$$

with

$$\lambda_{\gamma bk} = A_\beta(p^{1-\gamma}).$$

Moreover,  $\int_{\mathbb{Q}_p^n} \omega_{\gamma bk}(x) d^n x = 0$  and  $\{\omega_{\gamma bk}(x)\}_{\gamma bk}$  forms a complete orthogonal basis of  $L^2(\mathbb{Q}_p^n, d^n x)$ .

*Proof.* The result follows from Theorems 9.4.5 and 8.9.3 in [2], by using the fact that  $A_\beta$  satisfies  $A_\beta(\|p^\gamma(-p^{-1}k + \eta)\|_p) = A_\beta(\|p^{\gamma-1}k\|_p) = A_\beta(p^{1-\gamma})$ , for all  $\eta \in \mathbb{Z}_p^n$ .  $\square$

**Remark 5.2.** (i) Notice that  $\mathbf{A}_\beta$  has eigenvalues of infinity multiplicity. Now, if consider only eigenfunctions satisfying  $\text{supp } \omega_{\gamma bk}(x) \subset \mathbb{Z}_p^n$ , then necessarily  $\gamma \leq 0$  and  $b \in p^\gamma \mathbb{Z}_p^n / \mathbb{Z}_p^n$ . For  $\gamma$  fixed there are only a finite number of eigenfunctions  $\omega_{\gamma bk}$  satisfying  $\mathbf{A}_\beta \omega_{\gamma bk} = \lambda_{\gamma bk} \omega_{\gamma bk}$ , i.e. the multiplicities of the  $\lambda_{\gamma bk}$  are finite. Therefore we can number these eigenfunctions and eigenvalues in the form  $\omega_m, \lambda_m$  with  $m \in \mathbb{N} \setminus \{0\}$  such that  $\lambda_m \leq \lambda_{m'}$  for  $m \leq m'$ .

(ii) Notice that any  $\omega_m(x)$  is orthogonal to  $\Omega(\|x\|_p)$ , thus  $\{\omega_m(x)\}_{m \in \mathbb{N} \setminus \{0\}}$  is not a complete orthonormal basis of  $L^2(\mathbb{Z}_p^n, d^n x)$ . We now recall that  $\mathcal{L}_0(\mathbb{Z}_p^n)$  is dense in  $L_0^2(\mathbb{Z}_p^n)$ , and since the algebraic span of  $\{\omega_m(x)\}_{m \in \mathbb{N} \setminus \{0\}}$  contains  $\mathcal{L}_0(\mathbb{Z}_p^n)$ , then  $\{\omega_m(x)\}_{m \in \mathbb{N} \setminus \{0\}}$  is a complete orthonormal basis of  $L_0^2(\mathbb{Z}_p^n)$ .

**Proposition 5.3.** *Consider  $(\mathbf{A}_\beta, \mathcal{L}_0(\mathbb{Z}_p^n))$  and the eigenvalue problem:*

$$(5.2) \quad \mathbf{A}_\beta u = \lambda u, \quad \lambda > 0, \quad u \in \mathcal{L}_0(\mathbb{Z}_p^n).$$

Then the function  $u(x) = \omega_m(x)$  is a solution of (5.2) corresponding to  $\lambda = \lambda_m$ , for  $m \in \mathbb{N} \setminus \{0\}$ . In addition, the spectrum has the form

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots \text{ with } \lambda_m \uparrow +\infty,$$

where all the eigenvalues have finite multiplicity, and  $\{\omega_m(x)\}$ , with  $m \in \mathbb{N} \setminus \{0\}$ , is a complete orthonormal basis of  $L_0^2(\mathbb{Z}_p^n, d^n x)$ .

*Proof.* The result follows from Lemma 5.1, Remark 5.2 and (4.1).  $\square$

**Definition 5.4.** *We define the spectral zeta function attach to eigenvalue problem (5.2) as*

$$\zeta(s; \mathbf{A}_\beta, \mathcal{L}_0(\mathbb{Z}_p^n)) := \zeta(s; \mathbf{A}_\beta) = \sum_{m=1}^{\infty} \frac{1}{\lambda_m^s}, \quad s \in \mathbb{C}.$$

Later on we will show that  $\zeta(s; \mathbf{A}_\beta)$  converges if  $\text{Re}(s)$  is sufficiently big, and it does not depend on the basis  $\{\omega_m(x)\}$  used in its computation. By abuse of language (or following the classical literature, see [33]), we will say that  $\zeta(s; \mathbf{A}_\beta)$  is the spectral zeta function of operator  $\mathbf{A}_\beta$ .

5.1. **Example.** We compute  $\zeta(s; D_T^\beta)$ . We first note that

$$D_T^\beta \omega_{\gamma bk} = p^{-(\gamma-1)\beta} \omega_{\gamma bk}.$$

We now recall that if  $\text{supp } \omega_{\gamma bk} \subset \mathbb{Z}_p^n$  then  $\gamma \leq 0$  and  $b \in p^\gamma \mathbb{Z}_p^n / \mathbb{Z}_p^n$ . We now take  $-\gamma + 1 = m$ , with  $m \in \mathbb{N} \setminus \{0\}$ . Then  $b \in p^{-m+1} \mathbb{Z}_p^n / \mathbb{Z}_p^n$  and  $\lambda_m = p^{m\beta}$ , and the multiplicity of  $\lambda_m$  is equal to  $(p^n - 1)p^{n(m-1)} = p^{nm}(1 - p^{-n})$  for  $m \in \mathbb{N} \setminus \{0\}$ . Hence

$$\zeta(s; D_T^\beta) = \sum_{m=1}^{\infty} \frac{p^{nm}(1 - p^{-n})}{p^{m\beta s}} = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n \xi}{\|\xi\|_p^{\beta s}} = (1 - p^{-n}) \frac{p^{n-\beta s}}{1 - p^{n-\beta s}},$$

for  $\text{Re}(s) > \frac{n}{\beta}$ . Then  $\zeta(s; D_T^\beta)$  admits a meromorphic continuation to the whole complex plane as a rational function of  $p^{-s}$  with poles in the set  $\frac{n}{\beta} + \frac{2\pi i \mathbb{Z}}{\beta \ln p}$ .

## 6. HEAT TRACES AND $p$ -ADIC HEAT EQUATIONS ON THE UNTIL BALL

From now on,  $(\mathbf{A}_\beta, \text{Dom}(\mathbf{A}_\beta))$  is given by

$$(6.1) \quad (\mathbf{A}_\beta \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1}(\mathbf{A}_\beta(\xi) \mathcal{F}_{x \rightarrow \xi} \varphi) \text{ for } \varphi \in \text{Dom}(\mathbf{A}_\beta) = \mathcal{L}_0(\mathbb{Z}_p^n).$$

6.1.  **$p$ -adic heat equations on the until ball.** We introduce the following function:

$$K(x, t) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \chi_p(-x \cdot \xi) e^{-tA_\beta(\xi)} d^n \xi, \text{ for } t > 0, x \in \mathbb{Q}_p^n.$$

We note that by (4.1)  $e^{-tA_\beta(\xi)} \leq e^{-tC_0 \|\xi\|_p^\beta} \in L^1$  for  $t > 0$ , which implies that  $K(x, t)$  is well-defined for  $t > 0$  and  $x \in \mathbb{Q}_p^n$ .

**Lemma 6.1.** *With the above notation, the following formula holds:*

$$K(x, t) = \begin{cases} \Omega(\|x\|_p) \left\{ (1 - p^{-n}) \sum_{j=1}^{\text{ord}(x)} e^{-tA_\beta(p^j)} p^{nj} - p^{\text{ord}(x)n} e^{-tA_\beta(p^{\text{ord}(x)+1})} \right\} & \text{if } \text{ord}(x) \in \mathbb{N} \\ (1 - p^{-n}) \Omega(\|x\|_p) \sum_{j=1}^{\infty} e^{-tA_\beta(p^j)} p^{nj} & \text{if } \text{ord}(x) = +\infty \end{cases}$$

for any  $t > 0$ .

*Proof.* Take  $x = p^{\text{ord}(x)} x_0$ , with  $\|x_0\|_p = 1$ , then

$$\begin{aligned} K(x, t) &= \sum_{j=1}^{\infty} e^{-tA_\beta(p^j)} \int_{\|\xi\|_p = p^j} \chi_p(-x \cdot \xi) d^n \xi \\ &= \sum_{j=1}^{\infty} e^{-tA_\beta(p^j)} p^{nj} \int_{\|y\|_p = 1} \chi_p(-p^{-j+\text{ord}(x)} x_0 \cdot y) d^n y \\ &= \sum_{j=1}^{\infty} e^{-tA_\beta(p^j)} p^{nj} \begin{cases} 1 - p^{-n} & j \leq \text{ord}(x) \\ -p^{-n} & j = \text{ord}(x) + 1 \\ 0 & j \geq \text{ord}(x) + 2. \end{cases} \end{aligned}$$



Then  $K(x, t) = 0$  for  $\|x\|_p > 1$  and  $t > 0$ . Finally, we note that the announced formula is valid if  $x = 0$ .  $\square$

We identify  $L_0^2(\mathbb{Z}_p^n)$  with an isometric subspace of  $L^2(\mathbb{Q}_p^n)$  by extending the functions of  $L_0^2(\mathbb{Z}_p^n)$  as zero outside of  $\mathbb{Z}_p^n$ . We define  $\{T(t)\}_{t \geq 0}$  as the family of operators

$$\begin{aligned} L_0^2(\mathbb{Z}_p^n) &\rightarrow L_0^2(\mathbb{Z}_p^n) \\ f &\rightarrow T(t)f \end{aligned}$$

with

$$(T(t)f)(x) = \begin{cases} f(x) & \text{if } t = 0 \\ (K(\cdot, t) * f)(x) & \text{if } t > 0. \end{cases}$$

**Lemma 6.2.** *With the above notation the following assertions hold:*

- (i) *operator  $T(t)$ ,  $t \geq 0$ , is a well-defined bounded linear operator;*
- (ii) *for  $t \geq 0$ ,*

$$(T(t)f)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} \left[ 1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} \widehat{f}(\xi) \right],$$

*where  $\widehat{f}(\xi)$  denotes the Fourier transform in  $L^2(\mathbb{Q}_p^n)$  of  $f \in L_0^2(\mathbb{Z}_p^n)$ ;*

- (iii)  *$T(t)$ , for  $t > 0$ , is a compact, self-adjoint and non-negative operator.*

*Proof.* (i) We recall that  $K(\cdot, t) \in L^1(\mathbb{Q}_p^n)$  for  $t > 0$ , then, if  $f \in L_0^2(\mathbb{Z}_p^n) \subset L^2(\mathbb{Q}_p^n)$ , by the Young inequality,

$$u(x, t) := (K(\cdot, t) * f)(x) \in L^2(\mathbb{Q}_p^n), \text{ for } t > 0.$$

Now, by Lemma 6.1,  $\text{supp } u(x, t) \subset \mathbb{Z}_p^n$  for  $t > 0$ , i.e.  $u(x, t) \in L^2(\mathbb{Z}_p^n)$ , for  $t > 0$ . Again by the Young inequality,

$$\begin{aligned} \|u(x, t)\|_{L_0^2(\mathbb{Z}_p^n)} &= \|u(x, t)\|_{L^2(\mathbb{Q}_p^n)} \leq \|K(x, t)\|_{L^1(\mathbb{Q}_p^n)} \|f(x)\|_{L^2(\mathbb{Q}_p^n)} \\ &= C(t) \|f(x)\|_{L_0^2(\mathbb{Z}_p^n)}, \text{ for } t > 0. \end{aligned}$$

We finally show that

$$\int_{\mathbb{Z}_p^n} u(x, t) d^n x = 0, \text{ for } t > 0.$$

Indeed, for  $t > 0$ , by using Fubini's Theorem,

$$\begin{aligned} \int_{\mathbb{Z}_p^n} u(x, t) d^n x &= \int_{\mathbb{Z}_p^n} \left\{ \int_{\mathbb{Z}_p^n} K(y, t) f(x - y) d^n y \right\} d^n x \\ &= \int_{\mathbb{Z}_p^n} K(y, t) \left\{ \int_{\mathbb{Z}_p^n} f(x - y) d^n x \right\} d^n y \quad (\text{taking } z_1 = x - y, z_2 = y) \\ &= \int_{\mathbb{Z}_p^n} K(z_2, t) \left\{ \int_{\mathbb{Z}_p^n} f(z_1) d^n z_1 \right\} d^n z_2 = 0. \end{aligned}$$

(ii) Since  $f(x), u(x, t) \in L^1(\mathbb{Z}_p^n) \cap L^2(\mathbb{Z}_p^n)$  for  $t > 0$ , because  $L^2(\mathbb{Z}_p^n) \subset L^1(\mathbb{Z}_p^n)$ , we have

$$\mathcal{F}_{x \rightarrow \xi}(u(x, t)) = 1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} \widehat{f}(\xi),$$

this last function belongs to  $L^1(\mathbb{Q}_p^n)$ , indeed, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left\| 1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} \widehat{f}(\xi) \right\|_{L^1(\mathbb{Q}_p^n)} &\leq \left\| 1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} \right\|_{L^2(\mathbb{Q}_p^n)} \left\| \widehat{f}(\xi) \right\|_{L^2(\mathbb{Q}_p^n)} \\ &\leq \left\| e^{-tA_\beta(\xi)} \right\|_{L^2(\mathbb{Q}_p^n)} \|f(\xi)\|_{L^2(\mathbb{Q}_p^n)} = \left\| e^{-tA_\beta(\xi)} \right\|_{L^2(\mathbb{Q}_p^n)} \|f(\xi)\|_{L_0^2(\mathbb{Z}_p^n)} < \infty \end{aligned}$$

because  $\int_{\mathbb{Q}_p^n} e^{-2tA_\beta(\xi)} d^n \xi \leq \int_{\mathbb{Q}_p^n} e^{-2C_0 t \|\xi\|_p^\beta} d^n \xi < \infty$ , cf. (4.1). Finally,

$$\begin{aligned} (T(0)f)(x) &= \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \chi_p(-\xi \cdot x) \widehat{f}(\xi) d^n \xi \\ &= \int_{\mathbb{Q}_p^n} \chi_p(-\xi \cdot x) \widehat{f}(\xi) d^n \xi - \int_{\mathbb{Z}_p^n} \chi_p(-\xi \cdot x) \widehat{f}(\xi) d^n \xi \\ &= f(x) - \mathcal{F}_{x \rightarrow \xi}^{-1} \left( \Omega \left( \|\xi\|_p \right) \widehat{f}(\xi) \right) = f(x) - \Omega \left( \|x\|_p \right) * f(x) \\ &= f(x) - \Omega \left( \|x\|_p \right) \int_{\mathbb{Z}_p^n} f(x) d^n x = f(x). \end{aligned}$$

(iii) Since  $T(t)$ , for  $t > 0$ , is bounded and  $\langle T(t)f, g \rangle = \langle f, T(t)g \rangle$ , for  $f, g \in L^2(\mathbb{Z}_p^n)$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $L^2(\mathbb{Q}_p^n)$ ,  $T(t)$  is self-adjoint for  $t > 0$ . The compactness follows from the continuity of operator  $T(t)$ .  $\square$

**Lemma 6.3.** *The one-parameter family  $\{T(t)\}_{t \geq 0}$  of bounded linear operators from  $L_0^2(\mathbb{Z}_p^n)$  into itself is a contraction semigroup.*

*Proof.* The lemma follows from the following claims.

**Claim 1.**  $\|T(t)\|_{L_0^2(\mathbb{Z}_p^n)} \leq 1$  for  $t \geq 0$ . In addition,  $\|T(t)\|_{L_0^2(\mathbb{Z}_p^n)} < 1$  for  $t > 0$ .

Consider  $t > 0$ , by Lemma 6.2 and (4.1),

$$\begin{aligned} \|T(t)f\|_{L_0^2(\mathbb{Z}_p^n)}^2 &= \|T(t)f\|_{L^2(\mathbb{Q}_p^n)}^2 = \|\widehat{T(t)f(\xi)}\|_{L^2(\mathbb{Q}_p^n)}^2 = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-2tA_\beta(\xi)} |\widehat{f}(\xi)|^2 d^n \xi \leq \\ &\int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-2C_0 t \|\xi\|_p^\beta} |\widehat{f}(\xi)|^2 d^n \xi \leq \sup_{\xi \in \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-2C_0 t \|\xi\|_p^\beta} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} |\widehat{f}(\xi)|^2 d^n \xi \\ &< \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} |\widehat{f}(\xi)|^2 d^n \xi \leq \|f\|_{L^2(\mathbb{Q}_p^n)}^2 = \|f\|_{L_0^2(\mathbb{Z}_p^n)}^2, \end{aligned}$$

where we used that  $\sup_{\xi \in \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-2C_0 t \|\xi\|_p^\beta} < 1$ .

**Claim 2.**  $T(0) = I$ .

**Claim 3.**  $T(t+s) = T(t)T(s)$  for  $t, s \geq 0$ .

This claim follows from Lemma 6.2-(ii).

**Claim 4.** For  $f \in L_0^2(\mathbb{Z}_p^n)$ , the function  $t \rightarrow T(t)f$  belongs to  $C([0, \infty), L_0^2(\mathbb{Z}_p^n))$ .

Notice that since  $\mathcal{L}_0^2(\mathbb{Z}_p^n)$  is dense in  $L_0^2(\mathbb{Z}_p^n)$  for  $\|\cdot\|_{L^2}$  norm, it is sufficient to show Claim 4 for  $f \in \mathcal{L}_0^2(\mathbb{Z}_p^n)$ . Indeed,

$$\begin{aligned} \lim_{t \rightarrow t_0} \|T(t)f - T(t_0)f\|_{L_0^2(\mathbb{Z}_p^n)}^2 &= \lim_{t \rightarrow t_0} \|T(t)f - T(t_0)f\|_{L^2(\mathbb{Q}_p^n)}^2 \\ &= \lim_{t \rightarrow t_0} \left\| \widehat{T(t)f} - \widehat{T(t_0)f} \right\|_{L^2(\mathbb{Q}_p^n)}^2 = \lim_{t \rightarrow t_0} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} |\widehat{f}(\xi)|^2 |e^{-tA_\beta(\xi)} - e^{-t_0A_\beta(\xi)}|^2 d^n \xi, \end{aligned}$$

now, since  $1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) |\widehat{f}(\xi)|^2 |e^{-tA_\beta(\xi)} - e^{-t_0A_\beta(\xi)}|^2 \leq 4 |\widehat{f}(\xi)|^2$ , which is an integrable function, by applying the Dominated Convergence Theorem, we have  $\lim_{t \rightarrow t_0} \|T(t)f - T(t_0)f\|_{L_0^2(\mathbb{Z}_p^n)}^2 = 0$ .  $\square$

**Lemma 6.4.** *The infinitesimal generator of semigroup  $\{T(t)\}_{t \geq 0}$  restricted to  $\mathcal{L}_0(\mathbb{Z}_p^n)$  agrees with  $(-\mathbf{A}_\beta, \mathcal{L}_0(\mathbb{Z}_p^n))$ .*

*Proof.* We show that

$$\lim_{t \rightarrow 0^+} \left\| \frac{T(t)f - f}{t} + \mathbf{A}_\beta f \right\|_{L_0^2(\mathbb{Z}_p^n)} = 0, \text{ for } f \in \mathcal{L}_0(\mathbb{Z}_p^n).$$

Indeed, by Lemma 6.2-(ii),

$$\begin{aligned} \left\| \frac{T(t)f - f}{t} + \mathbf{A}_\beta f \right\|_{L_0^2(\mathbb{Z}_p^n)} &= \left\| \frac{T(t)f - f}{t} + \mathbf{A}_\beta f \right\|_{L^2(\mathbb{Q}_p^n)} \\ &= \left\| \frac{\widehat{T(t)f} - \widehat{f}}{t} + \widehat{\mathbf{A}_\beta f} \right\|_{L^2(\mathbb{Q}_p^n)} = \left\| \left\{ \frac{1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} - 1}{t} + A_\beta(\xi) \right\} \widehat{f}(\xi) \right\|_{L^2(\mathbb{Q}_p^n)}. \end{aligned}$$

Now we note that

$$\left\{ 1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} - 1 \right\} \widehat{f}(\xi) = \widehat{f}(\xi) \left\{ e^{-tA_\beta(\xi)} - 1 \right\} - 1_{\mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} \widehat{f}(\xi),$$

and since  $\text{supp } f \subset \mathbb{Z}_p^n$  then  $\widehat{f}(\xi + \xi_0) = \widehat{f}(\xi)$  for any  $\xi_0 \in \mathbb{Z}_p^n$ , this fact implies that  $1_{\mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} \widehat{f}(\xi) = e^{-tA_\beta(\xi)} \widehat{f}(0) = 0$  because  $f \in \mathcal{L}_0(\mathbb{Z}_p^n)$ . Hence

$$\begin{aligned} &\left\| \left\{ \frac{1_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n}(\xi) e^{-tA_\beta(\xi)} - 1}{t} + A_\beta(\xi) \right\} \widehat{f}(\xi) \right\|_{L^2(\mathbb{Q}_p^n)} \\ &= \left\| \frac{\{e^{-tA_\beta(\xi)} - 1\} \widehat{f}(\xi)}{t} + A_\beta(\xi) \widehat{f}(\xi) \right\|_{L^2(\mathbb{Q}_p^n)} \\ &= \left\| A_\beta(\xi) \widehat{f}(\xi) \left\{ 1 - e^{-\tau A_\beta(\xi)} \right\} \right\|_{L^2(\mathbb{Q}_p^n)} \quad (\text{for some } \tau \in (0, t)). \end{aligned}$$

Therefore, by the fact that  $A_\beta(\xi) \widehat{f}(\xi) \in \mathcal{L}_0(\mathbb{Q}_p^n)$  and the Dominated Convergence Theorem,

$$\lim_{t \rightarrow 0^+} \left\| \frac{T(t)f - f}{t} + \mathbf{A}_\beta f \right\|_{L_0^2(\mathbb{Z}_p^n)} = \lim_{t \rightarrow 0^+} \left\| A_\beta(\xi) \widehat{f}(\xi) \left\{ 1 - e^{-\tau A_\beta(\xi)} \right\} \right\|_{L^2(\mathbb{Q}_p^n)} = 0.$$

$\square$

**Theorem 6.5.** *The initial value problem:*

$$(6.2) \quad \begin{cases} u(x, t) \in C([0, \infty), \text{Dom}(\mathbf{A}_\beta)) \cap C^1([0, \infty), L_0^2(\mathbb{Z}_p^n)) \\ \frac{\partial u(x, t)}{\partial t} + \mathbf{A}_\beta u(x, t) = 0, \quad x \in \mathbb{Q}_p^n, \quad t \in [0, \infty) \\ u(x, 0) = \varphi(x) \in \text{Dom}(\mathbf{A}_\beta), \end{cases}$$

where  $(\mathbf{A}_\beta, \text{Dom}(\mathbf{A}_\beta))$  is given by (6.1) has a unique solution given by  $u(x, t) = T(t)\varphi(x)$ .

*Proof.* By Lemmas 6.3-6.4 and the Hille-Yosida-Phillips Theorem, see e.g. [8, Theorem 3.4.4], operator  $(-\mathbf{A}_\beta, \text{Dom}(\mathbf{A}_\beta))$  is  $m$ -dissipative with dense domain in  $L_0^2(\mathbb{Z}_p^n)$ , the announced theorem now follows from [8, Theorem 3.1.1 and Proposition 3.4.5].  $\square$

## 6.2. Heat Traces.

**Proposition 6.6.** *Let  $\{\omega_m\}_{m \in \mathbb{N} \setminus \{0\}}$  be the complete orthonormal basis of  $L_0^2(\mathbb{Z}_p^n)$  as above. Then*

$$K(x - y, t) = \sum_{m=1}^{\infty} e^{-\lambda_m t} \omega_m(x) \overline{\omega_m(y)}$$

where the convergence is uniform on  $\mathbb{Z}_p^n \times \mathbb{Z}_p^n \times [\epsilon, \infty)$ , for every  $\epsilon > 0$ .

*Proof.* By applying Hilbert-Schmidt Theorem to  $T(1)$ , see e.g. [25, Theorem VI.16], which is self-adjoint and compact, cf. Lemma 6.2 (iii), there exists a complete orthonormal basis  $\{\phi_m\}$ ,  $m \in \mathbb{N} \setminus \{0\}$ , of  $L_0^2(\mathbb{Z}_p^n)$  consisting of eigenfunctions of  $T(1)$ , let  $\{\mu_m\}$ ,  $m \in \mathbb{N} \setminus \{0\}$ , the sequence of corresponding eigenvalues. In addition,  $\mu_m \rightarrow 0$  as  $m \rightarrow \infty$ . By using the fact  $\{T(t)\}_{t \geq 0}$  form a semigroup,  $T(\frac{1}{k})\phi_m = \mu_m^{1/k} \phi_m$ , for every positive rational number  $\frac{1}{k}$ . Now from the continuity of  $\{T(t)\}_{t \geq 0}$  we get

$$T(t)\phi_m = \mu_m^t \phi_m, \text{ for } t \in \mathbb{R}_+.$$

We note that  $\mu_m > 0$  for every  $m$  since

$$\phi_m = \lim_{t \rightarrow 0^+} T(t)\phi_m = \phi_m \lim_{t \rightarrow 0^+} \mu_m^t$$

implies that  $\lim_{t \rightarrow 0^+} \mu_m^t = 1$  because  $\phi_m \neq 0$ . Hence  $\mu_m = e^{-\lambda_m}$ , with  $\lambda_m > 0$ , because  $\|T(t)\|_{L_0^2(\mathbb{Z}_p^n)} < 1$  for  $t > 0$ , cf. Lemma 6.3 (i), implies that  $\mu_m < 1$  and  $\lim_{m \rightarrow \infty} \lambda_m = \infty$ , since  $\lim_{m \rightarrow \infty} \mu_m = 0$ .

By using Mercer's Theorem, see e.g. [14, and the references therein], [26],

$$(6.3) \quad K(x - y, t) = \sum_{m=1}^{\infty} e^{-\lambda_m t} \phi_m(x) \overline{\phi_m(y)}.$$

Now, since  $T(t)\phi_m(x) = e^{-\lambda_m t} \phi_m(x)$  is a solution of problem (6.2) with initial datum  $\phi_m$ , cf. Theorem 6.5, and

$$-\lambda_m e^{-\lambda_m t} \phi_m(x) = \frac{\partial}{\partial t} (e^{-\lambda_m t} \phi_m(x)) = -\mathbf{A}_\beta (e^{-\lambda_m t} \phi_m(x)) = -e^{-\lambda_m t} \mathbf{A}_\beta \phi_m(x),$$

then  $\phi_m(x)$  is an eigenfunction of  $\mathbf{A}_\beta$  with  $\text{supp } \phi_m \subset \mathbb{Z}_p^n$ . Now, by using that  $\mathbf{A}_\beta \omega_m = \lambda_m \omega_m$ , see Proposition 5.3, we get that  $u = e^{-\lambda_m t} \omega_m$  is a solution of the following boundary value problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = -\mathbf{A}_\beta u(x,t), & u(x,t) \in L_0^2(\mathbb{Z}_p^n), \text{ for } t \geq 0 \\ u(x,0) = \omega_m(x), & \omega_m(x) \in \mathcal{L}_0(\mathbb{Z}_p^n). \end{cases}$$

Then, by Theorem 6.5, the above problem has a unique solution, which implies that

$$u(x,t) = T(t)\omega_m(x) = e^{-\lambda_m t} \omega_m,$$

hence we can replace  $\{\phi_m\}$  by  $\{\omega_m\}$  in (6.3).  $\square$

In the next result, we will use the classical notation  $e^{-t\mathbf{A}_\beta}$  for operator  $T(t)$  to emphasize the dependency on operator  $\mathbf{A}_\beta$ .

**Theorem 6.7.** *The operator  $e^{-t\mathbf{A}_\beta}$ , for  $t > 0$ , is trace class and it verifies:*

(i)

$$(6.4) \quad \text{Tr}(e^{-t\mathbf{A}_\beta}) = \sum_{m=1}^{\infty} e^{-\lambda_m t} = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-tA_\beta(\xi)} d^n \xi,$$

for  $t > 0$ ;

(ii) *there exist positive constants  $C, C'$  such that*

$$Ct^{-\frac{n}{\beta}} \leq \text{Tr}(e^{-t\mathbf{A}_\beta}) \leq C't^{-\frac{n}{\beta}},$$

for  $t > 0$ .

*Proof.* By Proposition 6.6 and the definition of  $K(x,t)$ :

$$(6.5) \quad K(0,t) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-tA_\beta(\xi)} d^n \xi = \sum_{m=1}^{\infty} e^{-\lambda_m t} |\omega_m(x)|^2,$$

for  $t > 0$ . By using the Dominated Convergence Theorem and the fact that  $\sum_m e^{-\lambda_m t}$  converges for  $t > 0$ , we can integrate both sides of (6.5) with respect to the variable  $x$  over  $\mathbb{Z}_p^n$  to get

$$(6.6) \quad \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-tA_\beta(\xi)} d^n \xi = \sum_{m=1}^{\infty} e^{-\lambda_m t}, \text{ for } t > 0.$$

We recall that

$$(6.7) \quad e^{-C_1 t \|\xi\|_p^\beta} \leq e^{-tA_\beta(\xi)} \leq e^{-C_0 t \|\xi\|_p^\beta},$$

cf. (4.1), and that  $e^{-Ct\|\xi\|_p^\beta} \in L^1$ , for  $t > 0$  and for any positive constant  $C$ , then the series on the right-hand side of (6.6) converges. Now

$$\begin{aligned} \text{Tr}(e^{-t\mathbf{A}_\beta}) &= \sum_{m=1}^{\infty} \langle e^{-t\mathbf{A}_\beta} \omega_m, \omega_m \rangle = \sum_{m=1}^{\infty} e^{-\lambda_m t} \|\omega_m\|_{L^2}^2 \\ &= \sum_{m=1}^{\infty} e^{-\lambda_m t} < \infty, \text{ for } t > 0, \end{aligned}$$

i.e.  $e^{-t\mathbf{A}_\beta}$  is trace class and the formula announced in (i) holds. The estimations for  $\text{Tr}(e^{-t\mathbf{A}_\beta})$  follows from (6.7), by using  $\int_{\mathbb{Q}_p^n} e^{-Ct\|\xi\|_p^\beta} d^n \xi \leq Dt^{-\frac{n}{\beta}}$  for  $t > 0$ .  $\square$

## 7. ANALYTIC CONTINUATION OF SPECTRAL ZETA FUNCTIONS

**Remark 7.1.** (i) We set for  $a > 0$ ,  $a^s := e^{s \ln a}$ . Then  $a^s$  becomes a holomorphic function on  $\operatorname{Re}(s) > 0$ .

(ii) We recall the following fact, see e.g. [17, Lemma 5.3.1]. Let  $(X, d\mu)$  denote a measure space,  $U$  a non-empty open subset of  $\mathbb{C}$ , and  $f : X \times U \rightarrow \mathbb{C}$  a measurable function. Assume that: (1) if  $\mathcal{C}$  is a compact subset of  $U$ , there exists an integrable function  $\phi_{\mathcal{C}} \geq 0$  on  $X$  satisfying  $|f(\xi, s)| \leq \phi_{\mathcal{C}}(\xi)$  for all  $(\xi, s) \in X \times \mathcal{C}$ ; (2)  $f(\xi, \cdot)$  is holomorphic on  $U$  for every  $x$  in  $X$ . Then  $\int_X f(\xi, s) d\mu$  is a holomorphic function on  $U$ .

**Proposition 7.2.** The spectral zeta function for  $\mathbf{A}_\beta$  is a holomorphic function on  $\operatorname{Re}(s) > \frac{n}{\beta}$  and satisfies

$$(7.1) \quad \zeta(s; \mathbf{A}_\beta) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n \xi}{A_\beta^s(\xi)} \text{ for } \operatorname{Re}(s) > \frac{n}{\beta}.$$

In particular  $\zeta(s; \mathbf{A}_\beta)$  does not depend on the basis of  $L_0^2(\mathbb{Z}_p^n)$  used in Definition 5.4.

*Proof.* By using Proposition 5.3 and Remark 5.2, the eigenvalues have the form  $A_\beta(p^{1-\gamma})$ , with  $\gamma \leq 0$ , and the corresponding multiplicity is the cardinality of  $p^\gamma \mathbb{Z}_p^n / \mathbb{Z}_p^n$  times the cardinality of the set of  $k$ 's, i.e.  $p^{-\gamma n} (p^n - 1)$ , then

$$\begin{aligned} \zeta(s; \mathbf{A}_\beta) &= \sum_{\gamma \leq 0} \frac{p^{-\gamma n} (p^n - 1)}{A_\beta^s(p^{1-\gamma})} = \sum_{m=1}^{\infty} \frac{p^{mn} (1 - p^{-n})}{A_\beta^s(p^m)} = \sum_{m=1}^{\infty} \int_{\|\xi\|_p = p^m} \frac{d^n \xi}{A_\beta^s(\|\xi\|_p)} \\ &= \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n \xi}{A_\beta^s(\xi)}, \end{aligned}$$

and by (4.1),

$$|\zeta(s; \mathbf{A}_\beta)| \leq \frac{(1 - p^{-n})}{C^{\operatorname{Re}(s)}} \sum_{m=1}^{\infty} p^{m(n - \beta \operatorname{Re}(s))} < \infty \text{ for } \operatorname{Re}(s) > \frac{n}{\beta}.$$

To establish the holomorphy on  $\operatorname{Re}(s) > \frac{n}{\beta}$  we use Remark 7.1 (ii). Take  $X = \mathbb{Q}_p^n \setminus \mathbb{Z}_p^n$ ,  $d\mu = d^n \xi$ ,  $U = \{s \in \mathbb{C}; \operatorname{Re}(s) > \frac{n}{\beta}\}$  and  $f(\xi, s) = A_\beta^{-s}(\|\xi\|_p)$ . We now verify the two conditions established in Remark 7.1 (ii). Take  $\mathcal{C}$  a compact subset of  $U$ , by (4.1)

$$\left| \frac{1}{A_\beta^s(\|\xi\|_p)} \right| \leq \frac{1}{C^{\operatorname{Re}(s)} \|\xi\|_p^{\beta \operatorname{Re}(s)}},$$

where  $C$  is a positive constant. Since  $\operatorname{Re}(s)$  belongs to a compact subset of

$$\left\{ s \in \mathbb{R}; \operatorname{Re}(s) > \frac{n}{\beta} \right\},$$

we may assume without loss of generality that  $\operatorname{Re}(s) \in [\gamma_0, \gamma_1]$  with  $\gamma_0 > \frac{n}{\beta}$ , then

$$\frac{1}{C^{\operatorname{Re}(s)} \|\xi\|_p^{\beta \operatorname{Re}(s)}} \leq B(\mathcal{C}) \frac{1}{\|\xi\|_p^{\beta \gamma_0}} \in L^1,$$

where  $B(\mathcal{C})$  is a positive constant. Condition (2) in Remark 7.1 (ii), follows from Remark 7.1 (i), by noting that  $(A_\beta(\|\xi\|_p))^{-s} = \exp(-s \ln A_\beta(\|\xi\|_p))$  with  $A_\beta(\|\xi\|_p) > 0$  for  $\|\xi\|_p > 1$ .  $\square$

**Remark 7.3.** We notice that formula (7.1) can be obtained by taking the Mellin transform in (6.4). Indeed,

$$\int_0^\infty \left\{ \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} e^{-t A_\beta(\|\xi\|_p)} t^{s-1} d^n \xi \right\} dt = \int_0^\infty \left\{ \sum_{m=1}^\infty e^{-\lambda_m t} t^{s-1} \right\} dt = \Gamma(s) \zeta(s; \mathbf{A}_\beta),$$

for  $\operatorname{Re}(s) > 1$ , where  $\Gamma(s)$  denotes the Archimedean Gamma function. Now, By changing variables as  $y = A_\beta(\|\xi\|_p)t$ , with  $\xi$  fixed, we have

$$\zeta(s; \mathbf{A}_\beta) = \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n \xi}{A_\beta^s(\|\xi\|_p)} \text{ for } \operatorname{Re}(s) > \max \left\{ 1, \frac{n}{\beta} \right\}.$$

**Lemma 7.4.**  $\zeta(s; \mathbf{A}_\beta)$  has a simple pole at  $s = \frac{n}{\beta}$ .

*Proof.* Set  $\sigma \in \mathbb{R}_+$ , since

$$\zeta(\sigma; \mathbf{A}_\beta) \leq \frac{1}{C_0} \int_{\mathbb{Q}_p^n \setminus \mathbb{Z}_p^n} \frac{d^n \xi}{\|\xi\|_p^{\beta\sigma}} = \frac{(1-p^{-n})p^{-\beta\sigma+n}}{C_0(1-p^{-\beta\sigma+n})} \text{ for } \sigma > \frac{n}{\beta},$$

we have

$$(7.2) \quad \lim_{\sigma \rightarrow \frac{n}{\beta}} (1 - p^{-\beta\sigma+n}) \zeta(\sigma; \mathbf{A}_\beta) > 0.$$

The assertion follows from (7.2), by using the fact that  $1 - p^{-\beta\sigma+n}$  has a simple zero at  $\frac{n}{\beta}$ . Indeed,

$$\begin{aligned} 1 - p^{-\beta\sigma+n} &= 1 - \exp\{(-\beta\sigma + n) \ln p\} \\ &= \{\beta \ln p\} \left( \sigma - \frac{n}{\beta} \right) + O\left( \left( \sigma - \frac{n}{\beta} \right)^2 \right), \end{aligned}$$

where  $O$  is an analytic function satisfying  $O(0) = 0$ .  $\square$

**Theorem 7.5.** The spectral zeta function  $\zeta(s; \mathbf{A}_\beta)$  satisfies the following:

- (i)  $\zeta(s; \mathbf{A}_\beta)$  is a holomorphic function on  $\operatorname{Re}(s) > \frac{n}{\beta}$ , and on this region is given by formula (7.1);
- (ii)  $\zeta(s; \mathbf{A}_\beta)$  has a simple pole at  $s = \frac{n}{\beta}$ , however, this pole is not necessarily unique;
- (iii) set  $N(T) := \sum_{\lambda_m \leq T} 1$ , for  $T \geq 0$ , then  $N(T) = O\left(T^{\frac{n}{\beta}}\right)$ .

*Proof.* (i) See Proposition 7.2. (ii) The first part was established in Lemma 7.4. Take  $\mathbf{A}_\beta$  to be the Taibleson operator  $D_T^\beta$ , then  $\zeta(s; \mathbf{D}_T^\beta)$  has a meromorphic continuation to the whole complex plane as a rational function of  $p^{-s}$  having poles in the set  $\frac{n}{\beta} + \frac{2\pi i \mathbb{Z}}{\beta \ln p}$ , see Example 5.1. (iii) The result follows from the formulas

$$\lambda_m = A_\beta(p^m) \text{ and } \operatorname{mult}(\lambda_m) = p^{nm} (1 - p^{-n}), \text{ for } m \in \mathbb{N} \setminus \{0\}.$$

$\square$

**Remark 7.6.** *The fact that  $\zeta(s; \mathbf{A}_\beta)$  may have several poles on the line  $\operatorname{Re}(s) = \frac{n}{\beta}$  prevent us of using the classical Ikehara Tauberian Theorem to obtain the asymptotic behavior of  $N(T)$ , see e.g. [11, Appendix A], [28, Chapter 2, Section 14]. Any way we expect that*

**Conjecture 7.7.**  *$N(T) \sim CT^{\frac{n}{\beta}}$ , for some suitable positive constant  $C$ .*

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